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# NOTES ON $\alpha$-BLOCH SPACE AND $D_p(\mu)$

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Abstract. In this paper, we will show that if $\mu$ is a Borel measure on the unit disk $D$ such that $\int_D \frac{d\mu(z)}{(1-|z|^2)^{p\alpha}} < \infty$ where $0 < \alpha, p < \infty$, then a bounded sequence of functions $\{f_n\}$ in the $\alpha$-Bloch space $\mathfrak{B}_\alpha$ has a convergent subsequence in the space $D_p(\mu)$ of analytic functions $f$ on $D$ satisfying $f' \in L^p(D, \mu)$. Also, we will find some conditions such that $\int_D \frac{d\mu(z)}{(1-|z|^2)^p} < \infty$.

## 1. Introduction

Let $D$ be the open unit disk in the complex plane $\mathbb{C}$. Let $H(D)$ be the space of all analytic functions on the unit disk $D$. The Bloch space $\mathfrak{B}$ of $D$ consists of analytic functions $f$ on $D$ such that

$$\sup\{(1-|z|^2)|f'(z)| : z \in D\} < \infty.$$

For $0 < \alpha < \infty$, a function $f \in H(D)$ is said to be in the $\alpha$-Bloch space $\mathfrak{B}_\alpha$ if

$$\| f \|_\alpha = \sup_{z \in D} |f'(z)|(1-|z|^2)^\alpha < \infty.$$

Let $\mu$ be a positive Borel measure on the unit disk $D$. For $0 < p < \infty$, we denote $D_p(\mu)$ as the space of analytic functions on $D$ satisfying

$$\| f \|_\mu = \left( \int_D |f'(z)|^p d\mu(z) \right)^{1/p} < \infty.$$

It is well known that inclusion map from $\mathfrak{B}$ to $D_p(\mu)$ is bounded if and only if $\int_D \frac{d\mu(z)}{(1-|z|^2)^p} < \infty$(See [2]). In this paper, we will investigate properties of some sequence in $\mathfrak{B}_\alpha$ related with $D_p(\mu)$ such that $\int_D \frac{d\mu(z)}{(1-|z|^2)^p} < \infty$.

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In this paper, we will use the symbol $C$ to denote a finite positive real number which does not depend on variables and may depend on some norms, not necessarily the same at each occurrence.

Let $N$ be the set of natural numbers. Let $\{f_n\}$ be a sequence in $\mathfrak{B}_\alpha$ such that $\sup\{\| f_n \|_\alpha : n \in N\} < C$ for some positive constant $C$. In section 2, we will show that if $\mu$ is the Borel measure on $D$ such that $\int_D \frac{d\mu(z)}{(1-|z|^2)^{p\alpha}} < \infty$, then $\{f_n\}$ has convergent subsequence in $D_p(\mu)$.

Let $r_n \in (0,1)$, $0 < \alpha \leq 1$ and $\lim_{n\to\infty} r_n = 1$. Let $f_{n,t}(z) = \sum_{k=1}^{\infty} a_k z^{2^k}$ where $a_k = 2^{k(\alpha-1)} {r_n}^{2^k-1} e^{i2^k t}$. Then $f_{n,t} \in \mathfrak{B}_\alpha$. In section 3, we will show that if $f_{n,t} \in D_{p\beta}(\mu)$ for $\beta > 1$ and $p > 0$, then $\int_D \frac{d\mu(z)}{(1-(r_n|z|)^2)^p} < \infty$. In particular, we will show that if $f_{n,t} \in D_{p\beta}(\mu)$ and $\| f_{n,t} \|_\mu < C$ where $C$ is independent of $n$ and $t$, then $\int_D \frac{d\mu(z)}{(1-|z|^2)^p} < \infty$.

## 2. Some sequence of functions in $\mathfrak{B}_\alpha$ which converge in $D_p(\mu)$

THEOREM 2.1. *Suppose that $f_n \in H(D)$ for $n \in N$ and if $\{f_n\}$ converges uniformly to $f$ on compact subsets of $D$. Then $f \in H(D)$ and $\{f_n'\}$ converges uniformly to $f'$ on compact subsets of $D$.*

*Proof.* See 10.28 Theorem in [9]. □

THEOREM 2.2. *If $K$ is compact, if $f_n \in C(K)$ for $n \in N$ and if $\{f_n\}$ is pointwise bounded and equicontinuous on $K$, then*

*(1) $\{f_n\}$ is uniformly bounded on $K$,*

*(2) $\{f_n\}$ contains a uniformly convergent subsequence.*

*Proof.* See 7.25 Theorem in [10]. □

LEMMA 2.3. *Let $\alpha > 0$ and $\{f_n\}$ be a bounded sequence in $\mathfrak{B}_\alpha$ such that $\sup\{\| f_n \|_\alpha : n \in N\} < C$ for some positive constant $C$. Then $\{f_n\}$ is equicontinuous on $D_r = \{z : |z| \leq r < 1\}$.*

*Proof.* For $z, w \in D_r$ and $0 \leq t \leq 1$,

$$|z + t(w-z)| = |(1-t)z + tw| \leq (1-t)|z| + t|w| \leq r.$$

For all $n \in N$,

$$\begin{aligned}
&|f_n(z) - f_n(w)| \\
&= \left| \int_0^1 f_n'(z + t(w - z))(w - z)dt \right| \\
&\leq |w - z| \int_0^1 |f_n'(z + t(w - z))| \frac{(1 - |z + t(w - z)|^2)^\alpha}{(1 - |z + t(w - z)|^2)^\alpha} dt \\
&\leq |w - z| \parallel f_n \parallel_\alpha \int_0^1 \frac{1}{(1 - |z + t(w - z)|^2)^\alpha} dt \\
&\leq C|w - z| \frac{1}{(1 - r^2)^\alpha}.
\end{aligned}$$

This implies that $\{f_n\}$ is equicontinuous on $D_r$. □

THEOREM 2.4. *Let* $\alpha > 0$ *and* $\{f_n\}$ *be a bounded sequence in* $\mathfrak{B}_\alpha$ *such that* $\sup\{\parallel f_n \parallel_\alpha : n \in N\} < C$ *for some constant* $C$. *Then there is a function* $f$ *in* $\mathfrak{B}_\alpha$ *such that some subsequence of* $\{f_n'\}$ *converges uniformly to* $f'$ *on compact subsets of* $D$ *and* $\parallel f \parallel_\alpha \leq C$.

*Proof.* For every $f_n \in \mathfrak{B}_\alpha$,

$$\begin{aligned}
|f_n(z) - f_n(0)| &= \left| \int_0^1 f_n'(tz)zdt \right| \\
&\leq \parallel f_n \parallel_\alpha \frac{1}{(1 - |z|^2)^\alpha} \\
&\leq C(1 - |z|^2)^{-\alpha}.
\end{aligned}$$

Thus $\{f_n\}$ is pointwise bounded and equicontinuous on compact subset $K$ of $D$. By Theorem 2.2, $\{f_n\}$ contains a uniformly convergent subsequence which converges to $f$. Without loss of generality, we may assume that the sequence $\{f_n\}$ itself converges to $f$. By Theorem 2.1, $\{f_n'\}$ converges to $f'$ on compact subsets of $D$.

Suppose that $\sup_{z \in D} f'(z)(1 - |z|^2)^\alpha > C + \epsilon$ for some $\epsilon > 0$. Then there exists $z_0$ such that $f'(z_0)(1 - |z_0|^2)^\alpha > C + \epsilon$. Since $f_n'(z_0)(1 - |z_0|^2)^\alpha \to f'(z_0)(1 - |z_0|^2)^\alpha$, we can find $n_0$ such that if $n > n_0$ then $f_n'(z_0)(1 - |z_0|^2)^\alpha > C + \epsilon/2$. This is a contradiction. Hence

$$\sup_{z \in D} f'(z)(1 - |z|^2)^\alpha \leq C.$$

This implies that $f \in \mathfrak{B}_\alpha$ and $\parallel f \parallel_\alpha \leq C$. □

THEOREM 2.5. *Let* $\alpha > 0$ *and* $\{f_n\}$ *be a sequence in* $\mathfrak{B}_\alpha$ *such that* $\sup\{\parallel f_n \parallel_\alpha : n \in N\} < C$ *for some constant* $C$. *Let* $\mu$ *be the positive*

*Borel measure on $D$ such that $\int_D \frac{d\mu(z)}{(1-|z|^2)^{p\alpha}} < \infty$ where $p > 0$. Then $\{f_n\}$ has convergent subsequence in $D_p(\mu)$.*

*Proof.* By the Theorem 2.4, $\{f_n'\}$ converges to $f'$ on every compact subset of $D$.

$$\begin{aligned} |f_n'(z) - f'(z)|^p &\le (2\max(|f_n'(z)|, |f'(z)|))^p \\ &\le 2^p(|f_n'(z)|^p + |f'(z)|^p) \\ &\le 2^p\left(|f_n'(z)|^p(1-|z|^2)^{p\alpha} + |f'(z)|^p(1-|z|^2)^{p\alpha}\right)(1-|z|^2)^{-p\alpha} \\ &\le 2^p(\| f_n \|_\alpha^p + \| f \|_\alpha^p)(1-|z|^2)^{-p\alpha} \\ &\le 2^p(C^p + C^p)(1-|z|^2)^{-p\alpha}. \end{aligned}$$

By the Lebesgue dominated convergence theorem,

$$\begin{aligned} \lim_{n\to\infty} \| f_n - f \|_\mu &= \left(\lim_{n\to\infty}\int_D |f_n'(z) - f'(z)|^p d\mu(z)\right)^{1/p} \\ &= \left(\int_D \lim_{n\to\infty} |f_n'(z) - f'(z)|^p d\mu(z)\right)^{1/p} \\ &= 0. \end{aligned}$$

□

## 3. Some condition for which $\int_D \frac{d\mu(z)}{(1-(|z|)^2)^p}$ is finite

Lemma 3.1. *For $0 < r < 1$ and $n \in N$,*

$$(2^n+1)(r^2)^{2^n} + \cdots + 2^{n+1}(r^2)^{2^{n+1}-1} \;<\; 2^{2n+1}r^{2^{n+1}}.$$

*Proof.* For all $n \in N$,

$$\begin{aligned} (2^n+1) + (2^n+2) + \cdots + 2^{n+1} &= \frac{2^{n+1}(2^{n+1}+1)}{2} - \frac{2^n(2^n+1)}{2} \\ &= 2^n(2^{n+1}+1) - 2^{n-1}(2^n+1). \end{aligned}$$

Since

$$2^{2n+1} - 2^n(2^{n+1}+1) + 2^{n-1}(2^n+1) = 2^{n-1}(2^n-1) > 0$$

and $0 < r < 1$,

$$(2^n+1)(r^2)^{2^n} + \cdots + 2^{n+1}(r^2)^{2^{n+1}-1} < 2^{2n+1}r^{2^{n+1}}$$

for all $n \in N$.

□

THEOREM 3.2. *For $0 < r < 1$ and $n \in N$,*

$$\frac{1}{(1-r^2)^2} \ < \ 2\left(1+\sum_{n=0}^{\infty} 2^{2n} r^{2^{n+1}}\right).$$

*Proof.* By Lemma 3.1,

$$\begin{aligned}
&1+\sum_{n=0}^{\infty} 2^{2n} r^{2^{n+1}}\\
&= \frac{1}{2}\left\{2+\sum_{n=0}^{\infty} 2^{2n+1} r^{2^{n+1}}\right\}\\
&> \frac{1}{2}\left\{2+\sum_{n=0}^{\infty}\left\{(2^n+1)(r^2)^{2^n}+\cdots+2^{n+1}(r^2)^{2^{n+1}-1}\right\}\right\}\\
&\geq \frac{1}{2}\left\{1+2r^2+3(r^2)^2+4(r^2)^3+5(r^2)^4+6(r^2)^5+\cdots\right\}\\
&= \frac{1}{2}\sum_{n=0}^{\infty}(n+1)(r^2)^n\\
&= \frac{1}{2}\sum_{n=0}^{\infty}\frac{\Gamma(n+2)}{n!\Gamma(2)}(r^2)^n\\
&= \frac{1}{2}\frac{1}{(1-r^2)^2}\ .
\end{aligned}$$

□

THEOREM 3.3. *Suppose that $\{n_k\}$ is an increasing sequence of positive integers such that $\frac{n_{k+1}}{n_k} \geq \lambda > 1$ for all $k$. Let $0 < p < \infty$. Then there is a positive constant $C$ depending on $p$ and $\lambda$ such that*

$$C^{-1}\left(\sum_{k=1}^{m}|a_k|^2\right)^{1/2} \leq \left(\frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{k=1}^{m} a_k e^{in_k\theta}\right|^p d\theta\right)^{1/p} \leq C\left(\sum_{k=1}^{m}|a_k|^2\right)^{1/2}$$

*for any scalars $a_1, a_2, \cdots, a_m$ and $m = 1, 2, \cdots$.*

*Proof.* See [11]. □

THEOREM 3.4. *Let $\{\lambda_n\}$ be a sequence of positive integers satisfying*

$$1 < \lambda \leq \frac{\lambda_{n+1}}{\lambda_n} \leq C < +\infty,\ n \geq 1.$$

*Suppose $\alpha > 0$ and $f(z) = \sum_{n=1}^{\infty} a_n z^{\lambda_n}$. Then $f \in \mathfrak{B}_\alpha$ if and only if $a_n = O(\lambda_n^{\alpha-1})$ as $n \to \infty$.*

*Proof.* See Theorem 21 in [12]. □

Let $\alpha > 0$, $r_n \in (0,1)$ and $\lim_{n\to\infty} r_n = 1$. Let $f_{n,t}(z) = \sum_{k=1}^{\infty} a_k z^{2^k}$ where $a_k = 2^{k(\alpha-1)} {r_n}^{2^k-1} e^{i2^k t}$. Since ${r_n}^{2^k-1} \to 0$ as $k \to \infty$, $f_{n,t} \in \mathfrak{B}_\alpha$ by Theorem 3.4.

THEOREM 3.5. *For $\beta > 1$ and $p > 0$,*

$$\left(\sum_{k=1}^{\infty} 2^{2k} (r_n|z|)^{2(2^k-1)\beta}\right)^{p/2} \leq C \frac{1}{2\pi} \int_0^{2\pi} |f'_{n,t}(z)|^{p\beta} dt$$

*for some constant $C$.*

*Proof.*

$$\begin{aligned} f'_{n,t}(z) &= \sum_{k=1}^{\infty} a_k 2^k z^{2^k-1} \\ &= \sum_{k=1}^{\infty} 2^{k/\beta} r_n^{2^k-1} e^{i2^k t} z^{2^k-1} \\ &= \sum_{k=1}^{\infty} 2^{k/\beta} (r_n z)^{2^k-1} e^{i2^k t}. \end{aligned}$$

By Theorem 3.3,

$$\begin{aligned} &\left(\frac{1}{2\pi}\int_0^{2\pi} \left|\sum_{k=1}^{\infty} 2^{k/\beta}(r_n z)^{2^k-1} e^{i2^k t}\right|^p dt\right)^{\beta} \\ &= \left(\frac{1}{2\pi}\int_0^{2\pi} \left|\sum_{k=1}^{\infty} 2^{k/\beta}(r_n z)^{2^k-1} e^{i2^k t}\right|^p dt\right)^{\frac{1}{p}p\beta} \\ &\geq C^{-p\beta} \lim_{m\to\infty} \left(\sum_{k=1}^{m} \left(2^{k/\beta}(r_n|z|)^{2^k-1}\right)^2\right)^{p\beta/2} \end{aligned}$$

This implies that

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi}|f'_{n,t}(z)|^{p\beta}dt\\
&=\left(\frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{k=1}^{\infty}2^{k/\beta}(r_n z)^{2^k-1}e^{i2^k t}\right|^{p\beta}dt\right)\left(\frac{1}{2\pi}\int_0^{2\pi}dt\right)\\
&\geq\left(\frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{k=1}^{\infty}2^{k/\beta}(r_n z)^{2^k-1}e^{i2^k t}\right|^{p}dt\right)^{\beta}\\
&\geq C^{-p\beta}\lim_{m\to\infty}\left(\sum_{k=1}^{m}\left(2^{k/\beta}(r_n|z|)^{2^k-1}\right)^2\right)^{p\beta/2}
\end{aligned}$$

where the first inequality follows from Hölder inequality. Since

$$\begin{aligned}
(x+y)^\beta &= (x+y)(x+y)^{\beta-1}\\
&= x(x+y)^{\beta-1}+y(x+y)^{\beta-1}\\
&\geq x^\beta+y^\beta
\end{aligned}$$

for all $x, y > 0$,

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi}|f'_{n,t}(z)|^{p\beta}dt\\
&\geq C^{-p\beta}\lim_{m\to\infty}\left(\sum_{k=1}^{m}\left(2^{k/\beta}(r_n|z|)^{2^k-1}\right)^2\right)^{p\beta/2}\\
&\geq C^{-p\beta}\lim_{m\to\infty}\left(\sum_{k=1}^{m}2^{2k}(r_n|z|)^{2(2^k-1)\beta}\right)^{p/2}\\
&= C^{-p\beta}\left(\sum_{k=1}^{\infty}2^{2k}(r_n|z|)^{2(2^k-1)\beta}\right)^{p/2}.
\end{aligned}$$

□

LEMMA 3.6. *If* $\beta > 1$ *and* $r_n \in (0,1)$, *then*

$$1-(r_n|z|)^2 \;<\; 1-(r_n|z|)^{2\beta} \;<\; C\left(1-(r_n|z|)^2\right)$$

*for some constant* $C$.

*Proof.* Since $r_n|z| < 1$, $1-(r_n|z|)^2 < 1-(r_n|z|)^{2\beta}$ .

$$\begin{aligned}
&\frac{1-(r_n|z|)^{2\beta}}{1-(r_n|z|)^2} = \frac{(1-(r_n|z|)^{\beta})(1+(r_n|z|)^{\beta})}{(1-(r_n|z|))(1+(r_n|z|))} \\
&\leq \frac{2(1-(r_n|z|)^{\beta})}{1-(r_n|z|)} \\
&\leq 2\frac{(1-(r_n|z|)^{[\beta]+1})}{1-(r_n|z|)} \\
&\leq 2\frac{(1-r_n|z|)(1+r_n|z|+\cdots+(r_n|z|)^{[\beta]})}{1-(r_n|z|)} \\
&\leq 2([\beta]+1).
\end{aligned}$$

Above two results imply that

$$1-(r_n|z|)^2 \;<\; 1-(r_n|z|)^{2\beta} \;<\; C\left(1-(r_n|z|)^2\right)$$

for $\beta > 1$ and $r_n \in (0,1)$. □

THEOREM 3.7. *Let $\beta > 1$ and $p > 0$. If $f_{n,t} \in D_{p\beta}(\mu)$, then*

$$\int_D \frac{d\mu(z)}{(1-(r_n|z|)^2)^p} \;<\; \infty.$$

*Proof.* By Theorem 3.2 and Lemma 3.6,

$$\begin{aligned}
&\sum_{k=1}^{\infty} 2^{2k}(r_n|z|)^{2(2^k-1)\beta} + 1 + (r_n|z|)^{2\beta} \\
&\geq 1 + \sum_{k=0}^{\infty} 2^{2k}(r_n|z|)^{2^{k+1}\beta} \\
&\geq \frac{1}{2}\frac{1}{(1-(r_n|z|)^{2\beta})^2} \\
&\geq C\frac{1}{(1-(r_n|z|)^2)^2}
\end{aligned}$$

for some constant $C$.

Above result and Theorem 3.5 imply that

$$\begin{aligned}&\frac{1}{(1-(r_n|z|)^2)^p}\\ &\leq C\left(\sum_{k=1}^{\infty} 2^{2k}(r_n|z|)^{2(2^k-1)\beta}\right)^{p/2}\\ &\leq C\frac{1}{2\pi}\int_0^{2\pi} |f'_{n,t}(z)|^{p\beta}dt.\end{aligned}$$

This implies that

$$\begin{aligned}&\int_D \frac{d\mu(z)}{(1-(r_n|z|)^2)^p}\\ &\leq \int_D C\frac{1}{2\pi}\int_0^{2\pi} |f'_{n,t}(z)|^{p\beta}dtd\mu(z)\\ &\leq C\frac{1}{2\pi}\int_0^{2\pi}\int_D |f'_{n,t}(z)|^{p\beta}d\mu(z)\ dt.\end{aligned}$$

□

Corollary 3.8. *If there is a constant $C$ such that $\int_D |f'_{n,t}(z)|^{p\beta}d\mu(z) \leq C$ where $C$ is a constant independent of $n$ and $t$, then $\int_D \frac{d\mu(z)}{(1-|z|^2)^p} < \infty$.*

*Proof.* This Corollary follows from Theorem 3.7 and Fatou's Lemma. □

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